Lesson 1—Coordinate Systems

Polar coordinates

Briggs–Cochran: Section 10.2, pages 645–646

The polar coordinate system is illustrated in Figure 1. A point P is specified by its distance r from the **origin** O (also called the **pole**) and the oriented angle θ formed by the ray $\overrightarrow{OP}$ and the **polar axis**, the ray labeled $\overrightarrow{OX}$ in the figure. A counterclockwise rotation from $\overrightarrow{OX}$ to $\overrightarrow{OP}$ gives a positive angle, and a clockwise rotation gives a negative angle. Every point corresponds to infinitely many pairs of polar coordinates, because the point is unchanged when the angle is altered by adding or subtracting a multiple of 2π.

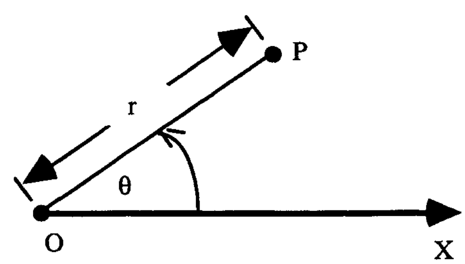

Figure 1: The Polar Coordinate System.

Points and sets of points can be plotted in polar coordinates using polar graph paper like that illustrated in Figure 2.

Usually when using polar coordinates, you will also be using a rectangular Cartesian coordinate system. In that case, the origins of the two coordinate systems are assumed to be the same point and the polar axis is assumed to be the positive x-axis. It is essential that you be able to switch back and forth between those two coordinate systems. Some simple trigonometry tells us

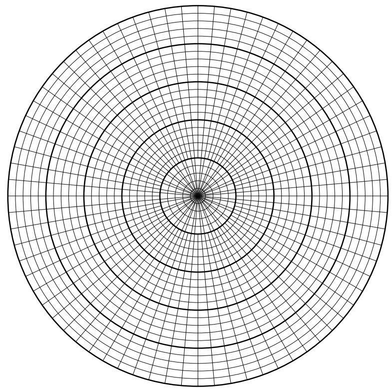

Figure 2: Polar Graph Paper.

that the conversion from polar coordinates to Cartesian coordinates is given by

$$\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,. \end{aligned}$$

To convert in the other direction, note that the Pythagorean theorem tells us

$$r = \sqrt{x^2 + y^2}\,.$$

Obtaining θ from x and y is not as clean and clear, because of the possibility of accidentally dividing by zero. As long as $x \neq 0$, we have $\tan\theta = y/x$, so

$$\theta = \begin{cases} \arctan(y/x)\,, & \text{if } (x, y) \text{ is in the first or fourth quadrant,} \\ \arctan(y/x) + \pi\,, & \text{if } (x, y) \text{ is in the second or third quadrant.} \end{cases}$$

The equations $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$ are important enough and used enough that you should memorize them, even if you hate memorization.

Cartesian coordinates in 3-space

Briggs–Cochran: pages 693–695

In the three-dimensional rectangular Cartesian coordinate system each point in space is represented by a triple of numbers (x, y, z) where the value of x is the signed distance to the yz-plane, the value of y is the signed distance to the xz-plane, and the value of z is the signed distance to the xy-plane. This is illustrated in Figure 3 for the point $(1, 2, 3)$.

Unless something else is specified, assume that the xy-plane is horizontal and the positive z axis points upward. It is customary to use a right-handed coordinate system like the one shown in Figure 3. With the positive z-axis pointing upward as usual and with you looking down on the xy-plane, in a right-handed coordinate system it will require a counterclockwise rotation to rotate the positive x-axis onto the positive y-axis.

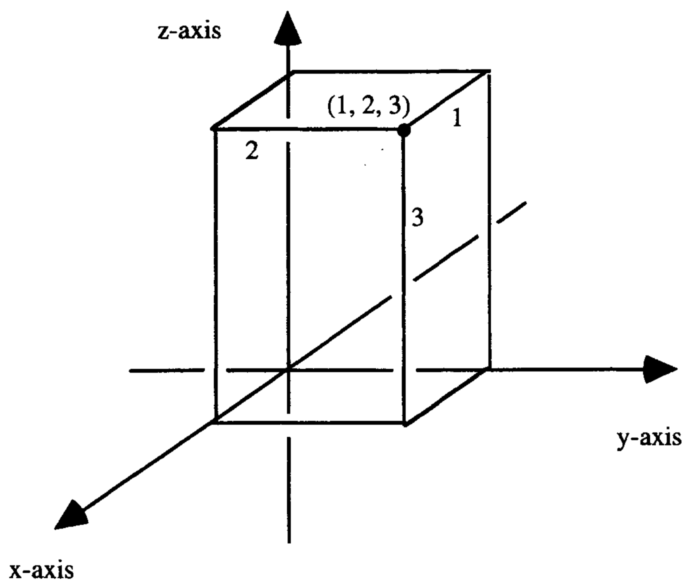

Figure 3: The Rectangular Cartesian Coordinate System.

Cylindrical coordinates

Briggs–Cochran: pages 918–920

The cylindrical coordinate system in 3-space uses polar coordinates in place of the x and y coordinates and leaves the third (usually vertical) coordinate z unaltered. This coordinate system is illustrated in Figure 4.

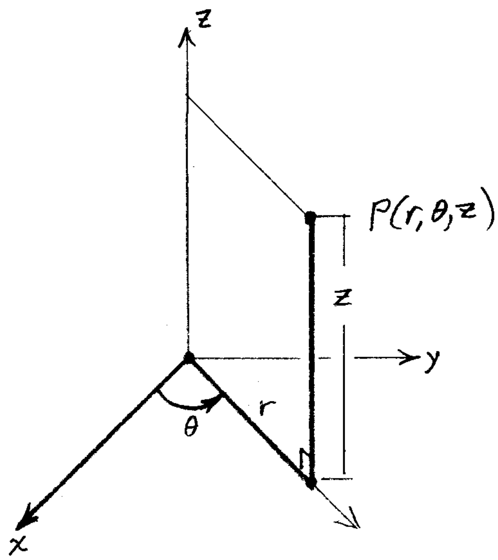

Figure 4: The Cylindrical Coordinate System.

Spherical coordinates

Briggs–Cochran: pages 924–926

The spherical coordinate system is often useful when dealing with a situation in which there is symmetry about a point in space. The rectangular coordinate system should be chosen so that the origin is the point of symmetry. One then converts between rectangular coordinates (x, y, z) and spherical coordinates (ρ, ϕ, θ) using the following formulas:

$$\begin{aligned} x &= \rho\ \sin\phi\ \cos\theta\,, \\ y &= \rho\ \sin\phi\ \sin\theta\,, \\ z &= \rho\ \cos\phi\,, \end{aligned}$$

and

$$\begin{aligned}\rho &= \sqrt{x^2+y^2+z^2}\,,\\ \tan\phi &= \sqrt{x^2+y^2}/z \quad \text{if } z \neq 0\,,\\ \tan\theta &= y/x \quad \text{if } x \neq 0\,.\end{aligned}$$

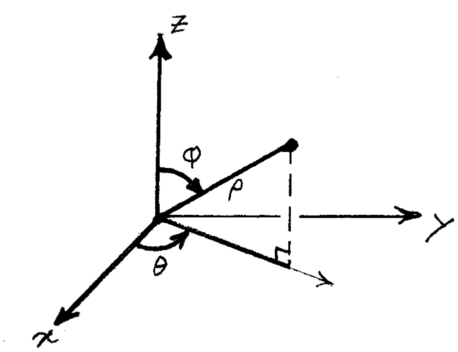

Figure 5: The Spherical Coordinate System.

The spherical coordinate ρ is the distance from the point to the origin. The angular coordinate ϕ in the spherical coordinate system is like the geographic coordinate of latitude, but ϕ is the angle measured downward from the North Pole, whereas the latitude is the angle north or south of the Equator. The angular coordinate θ in the spherical coordinate system agrees with the coordinate θ in the cylindrical coordinate system (so using the same symbol should be helpful rather than confusing). The spherical coordinate system is illustrated in Figure 5.

Listing the spherical coordinates in the order ρ, ϕ, θ is the convention used by American mathematicians, but it is not universal. More disturbingly, the names ϕ and θ are sometimes reversed, so check the definitions when using other sources.